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Stable core and chaos control in random Boolean networks

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Abstract. A variant of a simple method for chaos control is applied to achieve control in random Boolean networks (RBNs). It is shown that a RBN in the chaotic phase can be forced to behave periodically if a certain quenched fraction γ of the automata is given a fixed state (the system variables) every T time-steps. An analytic relationship between γ and T is derived using the size of the stable core as an order parameter.

A simple and paradigmatic example of complex systems has been provided by the random Boolean networks (RBNs), also called Kauffman nets [1–4, 20]. First introduced by Kauffman, a set of N binary elements $\boldsymbol{S}(t) = (S_1(t), \ldots, S_N(t))$, with $S_i(t) \in \Sigma \equiv \{0, 1\}$, $(i = 1, \ldots, N)$ is updated by means of the following dynamic equations $S_i(t+1) = \Lambda_i[S_{i_1}(t), S_{i_2}(t), \ldots, S_{i_K}(t)]$.

Such dynamical systems share some properties with cellular automata (CA), but here randomness is introduced at several levels. Each automaton is randomly connected with exactly K others which send inputs to it. Here Λ_i is a Boolean function also randomly chosen from a set $\mathcal{F}_K$ of all the Boolean functions with connectivity K. An additional source of randomness is introduced through the random choice of the initial condition $\boldsymbol{S}(0) \equiv \{S_i(0)\}$, drawn from the set $\mathcal{C}(N)$ of Boolean N-strings. In spite of these random choices, the RBNs exhibit a critical transition at $K_c = 2$. Two phases are observed: a frozen one, for $K < K_c$, and a chaotic phase for $K > K_c$ [3, 4]. Here 'chaos' is not the usual low-dimensional deterministic chaos but a phase where damage spreading takes place (i.e. propagation of changes caused by transient flips of a single unit).

This critical point was first estimated through numerical simulations [1, 2] and later analytically obtained by means of the so-called Derrida's annealed approximation (DAA) [5, 6, 18, 19]. Later on, Flyvbjerg [7] found analytically an order parameter, the size of the stable core at time infinity (defined below), for the second-order transition between the two phases. A simpler damage spreading approximation, equivalent to DAA, was introduced by Luque and Solé in [8].

Fogelman-Soulié [14, 15] defines the stable core as a set of variables of constant values through time and independent of initial conditions. On the other hand, Flyvbjerg [7] defines the stable core at time t as the variables that in time t have reached stable values (i.e. remain unaltered in value for $t' \geqslant t$ and are independent of initial conditions). He defines $s(t)$ as the relative size of the stable core at time t, i.e. $s(t)N$ is its absolute value and suggests the asymptotic stable core size, i.e. $s(t)$ when $t \to \infty$, as an order parameter for the second-order transition frozen chaos in RBNs. He obtains a growth iterated equation for the stable core (it is clear that $s(t)$ is non-decreasing)

$$s(t+1) = P(s(t)) \equiv \sum_{i=0}^{K} \binom{K}{i} s(t)^{K-i}(1-s(t))^{i} p_i \tag{1}$$

where p_i is the probability that the Boolean function is independent of a certain number i of inputs (a biologically relevant case are the canalizing functions that depend on a unique input and are independent of the rest [16]). For the Boolean functions with bias p, i.e. equal to 1 with probability p and equal to 0 with probability $(1-p)$ he finds

$$p_i = p^{2^i} + (1-p)^{2^i}. \tag{2}$$

By substituting (2) into (1) and analysing its stability he finds the transition critical curve

$$K2p(1-p) = 1 \tag{3}$$

according to [5, 6, 8], but reached by different methods.

In figure 1 we plot the relative asymptotic stable core size (full curve) for iteration of (1) and $K = 3$. Here we show that

$$\begin{aligned} s(\infty) &= 1 \qquad \text{if } p \geqslant p_{\rm c} \\ s(\infty) &< 1 \qquad \text{if } p < p_{\rm c} \end{aligned} \tag{4}$$

where $p_{\rm c}$ is exactly the critical value determined in (3) with $K = 3$. Thus (4) defines the ordered and chaotic or disordered phase, respectively.

We will show how to control the chaotic phase in a random Boolean network by means of periodic pulses in the system variable from the point of view of the stable core. In recent years, chaos control [9] has been widely used in the analysis of many dynamical systems and biological implications have been suggested [10]. This theory has been successfully applied to different real systems [9]. Recent developments in molecular genetics allow us to modify the activity of single genes and in so doing open the door for control mechanisms. We will use a variant of the Güémez and Matías (GM) method [11]. This simple way of controlling chaos has been successfully applied to n-dimensional maps, to discrete neural networks [12] and in the control of spatiotemporal chaos in coupled map lattice models [13]. Our aim in this paper is to retrieve former results [17] on chaos control in RBNs using the growth equation (1) for the stable core and to analyse the outcome from this point of view. We wish to 'push' the system from the chaotic to the ordered phase. In order to reach this point, we will help the stable core to grow sufficiently to reach the whole system. A simple method can be used: we will periodically freeze (with period T) a fraction γ within the whole set of variables [17]. The fraction γ will be randomly chosen, but, once determined, it will remain fixed (quenched). Similarly with the values established for the fraction γ, these will be randomly chosen and maintained afterwards (quenched). In figure 2 we illustrate the application of this method for the particular case of a RBN with $K = 3$ and $p = 0.5$. The plot shows the magnetization or activity of the network in time, defined as $M(t) = (1/N)\sum S_i(t)$, where N is the total number of automata ($N = 1000$ in this case). The system appears to be chaotic for $t = 0$–100, when the pulse-control is applied, consisting of a frozen fraction $\gamma = 0.6$ of the total N every $T = 2$ time-steps. For $t = 100$–200 the control pushes the system to become ordered. For $t = 200$ the control is removed and the system becomes chaotic again. We can define the magnetization of an automata as the temporal average of its states [19]. In the ordered regime, the histogram of the observed magnetizations over the entire network is a series of discrete peaks (see top middle figure). On the other hand, in the chaotic regime, this histogram is continuous (see top left and right figures).

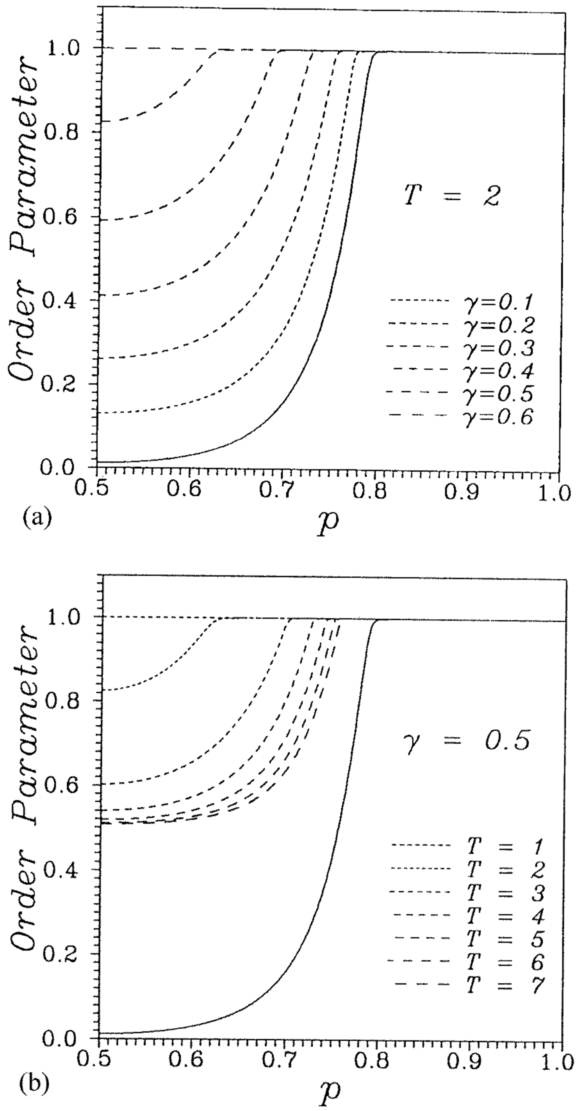

Figure 1. (a) Critical lines (for $K = 3$) obtained from iterating equation (5), here a fixed period T is used and different quenched fractions γ are applied; (b) same as before, but now $\gamma = 0.5$ is fixed and different periods are applied (see text).

External controls force us to rewrite equation (1), developed for an autonomous system. It is now a system that receives a perturbation of size γ each T time-steps

$$s(t+i) = P(s(t)) \qquad i = 1, 2, \ldots, T-1 \tag{5a}$$

$$s(t+T) = \gamma + (1-\gamma)P(s(t+T-1)). \tag{5b}$$

Equation (5*b*) is the unique novelty with respect to (1). Here we observe that the γ percentage, artificially frozen, helps the stable core to grow, and how, over the remaining fraction $(1-\gamma)$, the same mechanism operates. Again, for the system, $s^* = 1$ is a fixed

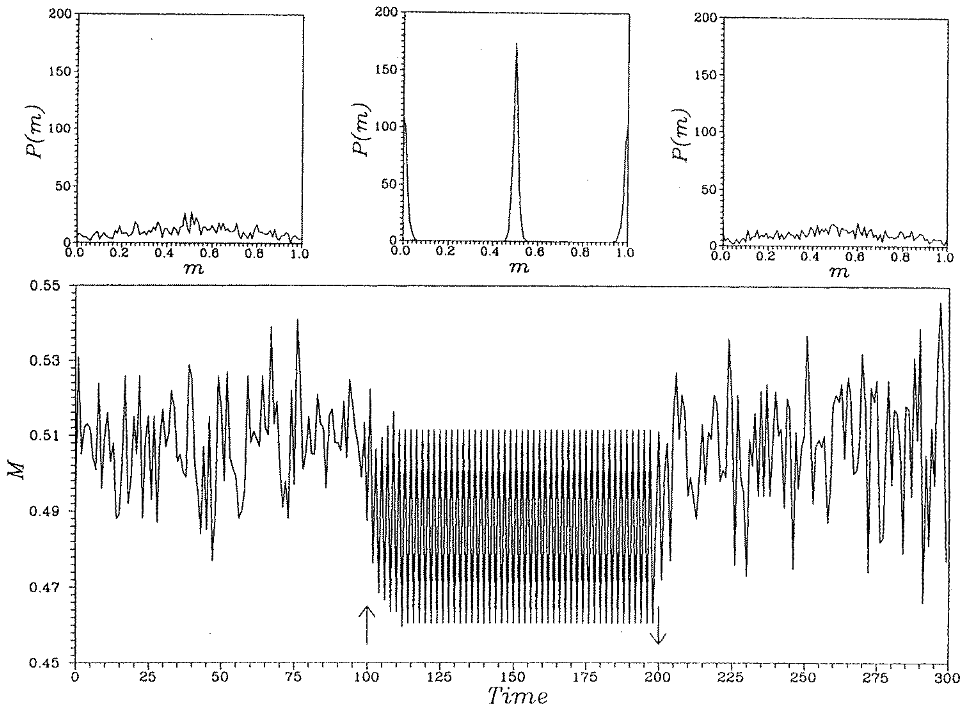

Figure 2. Control of a chaotic RBN. Magnetization or activity in time of a system with $N = 1000$ automata, connectivity $\langle k \rangle = 3$ and bias $p = 0.5$ (thus in the chaotic phase) is displayed. The control has been applied at $t = 100$ with $\gamma = 0.6$ and period $T = 2$. The network reaches the stable state in a few time-steps, until the control is removed at $t = 200$.

point and we can analyse its stability through the inequality

$$\left.\frac{\partial s(t+T)}{\partial s(t)}\right|_{s^*} < 1. \tag{6}$$

By using the chain rule, we obtain as a critical condition for stability

$$2Kp(1-p)(1-\gamma)^{1/T} = 1 \tag{7}$$

which is equal to previously obtained results [17]. Our results are summarized for $K = 3$ in figure 1(a) and (b). In both figures the critical lines have been obtained through iteration of the dynamical system (5). In figure 1(a) we show the order parameter for a fixed T and different γ-values. At a given (large enough) γ, all the elements belong to the stable core. In figure 1(b), for a fixed γ we change the period T. For $T = 1$ and $\gamma = 0.5$, half of the units are always frozen and the stable core grows since the whole net becomes frozen. We see that for $T = 2$ and $T = 3$ a strong reduction of the stable core is obtained. Further increases in T lead to an asymptotic approach of the order parameter towards $s(\infty) = \gamma = 0.5$ for $p = 0.5$ and to the critical point $p_c = 0.79$ as expected from the limit $T \to \infty$ of equation (7).

To sum up, the stable core approach to RBNs enables us to describe in a quantitative fashion the conditions for controlling chaotic dynamics (in the sense of a RBN) under periodic quenched network perturbations. Moreover, an exact relation between the network parameters and the control parameters can be derived with remarkable similarities using continuous dynamical systems [17].

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